

AP Calculus AB
Test: Inverse and Transcendental Functions
Instructor's Version

Free-Response Question 1 — no calculator (15 points)

1. Given that $f(x) = \arctan x$ and $g(x) = e^{\left(\frac{1}{2}\right)^{x-1}}$:

A. Write the equation of the line tangent to $f(x)$ at $x = \sqrt{3}$. (3 points)

Answer:

$$f'(x) = \frac{1}{1+x^2}$$

$$f'(\sqrt{3}) = \frac{1}{4}$$

$$f(\sqrt{3}) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$y - \frac{\pi}{3} = \frac{1}{4}(x - \sqrt{3})$$

1 point: $f'(\sqrt{3}) = \frac{1}{4}$

1 point: $f(\sqrt{3}) = \frac{\pi}{3}$

1 point: equation of line

B. Write the equation of the normal line to $f(x)$ at $x = 1$. (4 points)

Answer:

$$f'(x) = \frac{1}{2}; \text{ slope of normal line is } 2$$

$$f(1) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$y - \frac{\pi}{4} = 2(x - 1)$$

1 point: slope = 2

1 point: $f(1) = \frac{\pi}{4}$

2 points: equation of line

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- C. Find the area enclosed by the x -axis, the function $g(x)$, and the vertical lines $x = 2$ and $x = 4$. (4 points)

Answer:

$$A = \int_2^4 e^{\left(\frac{1}{2}\right)^{x-1}} dx, \text{ let } u = \frac{x}{2} - 1, \quad du = \frac{1}{2} dx$$
$$= 2 \int_0^1 e^u du = 2e^u \Big|_0^1 = 2e - 2$$

1 point: integrand with limits
2 points: antiderivative
1 point: answer

- D. Find the area enclosed by the x -axis, the function $f^{-1}(x)$, and the vertical lines $x = 0$ and $x = \frac{\pi}{3}$. (4 points)

Answer:

$$A = \int_0^{\frac{\pi}{3}} \tan x = \ln|\sec x| \Big|_0^{\frac{\pi}{3}} = \ln 2 - \ln 1 = \ln 2$$

1 point: integrand
2 points: antiderivative
1 point: answer

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Free-Response Question 2 — with calculator (15 points)

2. Consider the exponential function $f(x) = 2^{x-2} + 1$.

- A. Find the inverse function, $f^{-1}(x)$, and state the domain of the inverse function.
(4 points)

Answer:

Using the method of swapping x's and y's and solving for y:

$$x = 2^{y-2} + 1$$

$$2^{y-2} = x - 1$$

$$\ln 2^{y-2} = \ln(x - 1)$$

$$y - 2 = \frac{\ln(x - 1)}{\ln 2}$$

$$f^{-1}(x) = \frac{\ln(x - 1)}{\ln 2} + 2$$

alternate forms:

$$f^{-1}(x) = \begin{cases} \frac{\log(x-1)}{\log 2} + 2 \\ \log_2(x-1) + 2 \end{cases}$$

Domain of the inverse function is $(1, \infty)$.

1 point: set-up $x = 2^{y-2} + 1$

1 point: takes logarithm $\ln 2^{y-2} = \ln(x - 1)$

1 point: writes inverse function $f^{-1}(x) = \frac{\ln(x - 1)}{\ln 2} + 2$

1 point: indicates domain

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- B. Find the coordinates (x, y) of the two points of intersection between $f(x)$ and the line $y = x$. (2 points)

Answer:

Curves intersect where $2^{x-2} + 1 = x$.

Solutions are $x = 2$ and $x = 3$, so the two points of intersection are $(2, 2)$ and $(3, 3)$.

2 points: 1 point for each intersection point.

- C. Determine the exact area of the region bounded by $f(x)$ and $y = x$. (3 points)

Answer:

$$A = \int_2^3 (x - 2^{x-2} - 1) dx$$

$$= \left[\frac{x^2}{2} - \frac{2^{x-2}}{\ln 2} - x \right]_2^3$$

$$= \frac{9}{2} - \frac{2}{\ln 2} - 3 - \left(\frac{4}{2} - \frac{1}{\ln 2} - 2 \right)$$

$$= \frac{3}{2} - \frac{1}{\ln 2}$$

1 point: set up of definite integral $A = \int_2^3 (x - 2^{x-2} - 1) dx$

2 points: evaluates integral and gives answer as exact value $= \frac{3}{2} - \frac{1}{\ln 2}$

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- D. Determine the area of the region bounded by $f(x)$ and $f^{-1}(x)$. (2 points)

Answer:

Since $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = x$, the area of the region is twice that in part (C).

$$A = 3 - \frac{2}{\ln 2}$$

1 point: answer

1 point: justification

- E. Determine the value of a , such that the average value of the function $f(x)$ on the interval $[2, a]$ is 8. Round your value for a to 3 decimal places. (4 points)

Answer:

$$8 = \frac{1}{a-2} \int_2^a (2^{x-2} + 1) dx$$

$$8a - 16 = \frac{2^{x-2}}{\ln 2} + x \Big|_2^a$$

$$8a - 16 = \frac{2^{a-2}}{\ln 2} + a - \frac{1}{\ln 2} - 2 \text{ or equivalent equation to solve with calculator}$$

$$2^{a-2} + (14 - 7a)\ln 2 - 1 = 0$$
$$a \approx 6.519$$

$$2 \text{ points: set-up } 8 = \frac{1}{a-2} \int_2^a (2^{x-2} + 1) dx$$

$$1 \text{ point: equation to solve } 2^{a-2} + (14 - 7a)\ln 2 - 1 = 0$$

$$1 \text{ point: solution } a \approx 6.519$$